

# Set Theory Exercises

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Introduction to Mathematical Thinking

## Exercises 1

**Answer 1** The set  $\{n \in \mathbb{N} \mid (n > 1) \wedge (\forall x, y \in \mathbb{N})[(xy = n) \implies (x = 1 \vee y = 1)]\}$  is the set of all prime numbers.

**Answer 2**  $P = \{x \in \mathbb{R} \mid \sin(x) = 0\} = \{n\pi \mid n \in \mathbb{Z}\} = Q$ . Therefore, the relationship between  $P$  and  $Q$  is that  $P = Q$ .

**Answer 3** A simpler definition of the set  $A = \{x \in \mathbb{R} \mid (x > 0) \wedge (x^2 = 3)\}$  is  $A = \{\sqrt{3}\}$ .

**Answer 4 Proof that  $\emptyset \subseteq A$ :** By definition, a set  $X$  is a subset of  $Y$  if for all  $x$ ,  $x \in X \implies x \in Y$ . Let  $x$  be any element and  $A$  be any set. We wish to show that  $x \in \emptyset \implies x \in A$ . This conditional statement is vacuously true because the antecedent  $x \in \emptyset$  is always false. Therefore, for any set  $A$ ,  $\emptyset \subseteq A$ .

**Proof that  $A \subseteq A$ :** Let  $A = \{x \mid P(x)\}$  for some property  $P$ . Pick any  $x \in A$ . Since the left and right sides refer to the same set  $A$ , any element  $x$  in  $A$  on the left side is trivially in  $A$  on the right side ( $x \in A \implies x \in A$ ). Therefore,  $A \subseteq A$ .

**Answer 5 Proof:** Assume  $A \subseteq B$  and  $B \subseteq C$ . This means that for all  $x \in A$ ,  $x \in B$ , and for all  $y \in B$ ,  $y \in C$ .

Let  $x_a$  be any arbitrary element of  $A$ . By our first assumption ( $A \subseteq B$ ),  $x_a \in B$ . By our second assumption ( $B \subseteq C$ ), since  $x_a \in B$ , it follows that  $x_a \in C$ .

Since any arbitrary element  $x_a \in A$  is also in  $C$ , we have  $A \subseteq C$ . Therefore, if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Answer 6** The set  $\{1, 2, 3, 4\}$  has  $2^4 = 16$  subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$$

**Answer 7** The set  $\{1, 2, 3, \{1, 2\}\}$  has  $2^4 = 16$  subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{\{1, 2\}\}, \{1, 2\}, \{1, 3\}, \{1, \{1, 2\}\}, \{2, 3\}, \{2, \{1, 2\}\}, \\ \{3, \{1, 2\}\}, \{1, 2, 3\}, \{1, 2, \{1, 2\}\}, \{1, 3, \{1, 2\}\}, \{2, 3, \{1, 2\}\}, \{1, 2, 3, \{1, 2\}\}$$

**Answer 8 Proof by Contradiction:** We want to show that for all  $x$ , if  $x \in A$ , then  $x \in B$ .

Fix any element  $x \in A$ , which means  $x$  satisfies  $P(x)$ . Assume for contradiction that  $x \notin B$ , which means  $x$  does not satisfy  $Q(x)$ .

However,  $P$  and  $Q$  are formulas such that  $\forall x[P(x) \implies Q(x)]$ . Having  $P(x)$  be true while  $Q(x)$  is false makes the conditional  $P(x) \implies Q(x)$  false, contradicting the premise  $\forall x[P(x) \implies Q(x)]$ .

Therefore, by contradiction,  $x \in B$ . Hence,  $A \subseteq B$ .

**Answer 9 Proof by Induction:** We prove that any set with  $n$  elements has  $2^n$  subsets for all  $n \geq 0$ .

*Base Case* ( $n = 0$ ): The empty set  $\emptyset$  has 0 elements. Its only subset is  $\emptyset$ , so it has  $1 = 2^0$  subset. The claim holds for  $n = 0$ .

*Inductive Step:* Assume that the statement holds for all sets with cardinality  $n$ , meaning any set with  $n$  elements has  $2^n$  subsets. We wish to show that any set  $C$  with cardinality  $n + 1$  has  $2^{n+1}$  subsets.

Let  $C$  be a set with  $n + 1$  elements. Choose an element  $x \in C$  and let  $A = C \setminus \{x\}$ , so  $A$  has  $n$  elements.

A subset  $S \subseteq C$  either does not contain  $x$  or contains  $x$ :

- If  $x \notin S$ , then  $S$  is a subset of  $A$ . By the induction hypothesis, there are  $2^n$  such subsets.
- If  $x \in S$ , then  $S = T \cup \{x\}$  for some  $T \subseteq A$ . By the induction hypothesis, there are  $2^n$  choices for  $T$ , and hence  $2^n$  such subsets.

Thus, the total number of subsets of  $C$  is  $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ .

By the principle of mathematical induction, a set with exactly  $n$  elements has  $2^n$  subsets for all  $n \geq 0$ .

**Answer 10** An alternative definition of  $A = \{o, t, f, s, e, n\}$  is:

$A = \{x \mid x \text{ is the initial letter of the English word for a natural number } n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$

(corresponding to **o**ne, **t**wo, **t**hree, **f**our, **f**ive, **s**ix, **s**even, **e**ight, **n**ine).

## Exercises 2: Proofs of Set Operation Properties

Let  $A, B, C$  be subsets of a universal set  $U$ .

**(1) Associativity of Union:**  $A \cup (B \cup C) = (A \cup B) \cup C$  *Proof:*

- ( $\subseteq$ ): Let  $x \in A \cup (B \cup C)$ . By definition,  $(x \in A) \vee (x \in B \cup C)$ .
  - If  $x \in A$ , then  $x \in A \cup B$ , and consequently  $x \in (A \cup B) \cup C$ .
  - If  $x \in B \cup C$ , then  $(x \in B) \vee (x \in C)$ . If  $x \in B$ , then  $x \in A \cup B$ , so  $x \in (A \cup B) \cup C$ . If  $x \in C$ , then  $x \in (A \cup B) \cup C$ .

In all cases, if  $x \in A \cup (B \cup C)$ , then  $x \in (A \cup B) \cup C$ .

- ( $\supseteq$ ): Let  $x \in (A \cup B) \cup C$ . By definition,  $(x \in A \cup B) \vee (x \in C)$ .
  - If  $x \in A \cup B$ , then  $(x \in A) \vee (x \in B)$ . If  $x \in A$ , then  $x \in A \cup (B \cup C)$ . If  $x \in B$ , then  $x \in B \cup C$ , so  $x \in A \cup (B \cup C)$ .
  - If  $x \in C$ , then  $x \in B \cup C$ , so  $x \in A \cup (B \cup C)$ .

In all cases, if  $x \in (A \cup B) \cup C$ , then  $x \in A \cup (B \cup C)$ .

Therefore,  $A \cup (B \cup C) = (A \cup B) \cup C$ .

**(2) Associativity of Intersection:**  $A \cap (B \cap C) = (A \cap B) \cap C$  *Proof:*

- ( $\subseteq$ ): Let  $x \in A \cap (B \cap C)$ . By definition of intersection,  $(x \in A) \wedge (x \in B \cap C)$ . Since  $x \in B \cap C$ , we have  $x \in B$  and  $x \in C$ . Thus,  $x$  is in  $A$ ,  $B$ , and  $C$ . Since  $x \in A$  and  $x \in B$ ,  $x \in A \cap B$ . Together with  $x \in C$ , we have  $x \in (A \cap B) \cap C$ .
- ( $\supseteq$ ): Let  $x \in (A \cap B) \cap C$ . By definition,  $(x \in A \cap B) \wedge (x \in C)$ . Since  $x \in A \cap B$ , we have  $x \in A$  and  $x \in B$ . Thus,  $x$  is in  $A$ ,  $B$ , and  $C$ . Since  $x \in B$  and  $x \in C$ ,  $x \in B \cap C$ . Together with  $x \in A$ , we have  $x \in A \cap (B \cap C)$ .

Therefore,  $A \cap (B \cap C) = (A \cap B) \cap C$ .

**(3) Commutativity of Union:**  $A \cup B = B \cup A$  *Proof:*

- ( $\subseteq$ ): Let  $x \in A \cup B$ . By definition of union,  $x \in A$  or  $x \in B$ . If  $x \in A$ , then  $x \in B \cup A$ . If  $x \in B$ , then  $x \in B \cup A$ .
- ( $\supseteq$ ): Let  $x \in B \cup A$ . By definition of union,  $x \in B$  or  $x \in A$ . If  $x \in B$ , then  $x \in A \cup B$ . If  $x \in A$ , then  $x \in A \cup B$ .

Therefore,  $A \cup B = B \cup A$ .

**(4) Commutativity of Intersection:**  $A \cap B = B \cap A$  *Proof:* Let  $x \in A \cap B$ . By definition of intersection,  $x \in A$  and  $x \in B$ . Since logical conjunction is commutative,  $x \in B$  and  $x \in A$ , which by definition means  $x \in B \cap A$ . By the same reasoning, any  $x \in B \cap A$  is also in  $A \cap B$ .

Therefore,  $A \cap B = B \cap A$ .

**(5) Distributivity of Union over Intersection:**  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  *Proof:*

- $(\subseteq)$ : Let  $x \in A \cup (B \cap C)$ . By definition of union,  $x \in A$  or  $x \in B \cap C$ .
  - If  $x \in A$ , then  $x \in A \cup B$  and  $x \in A \cup C$  by definition of union. Thus,  $x \in (A \cup B) \cap (A \cup C)$ .
  - If  $x \in B \cap C$ , then  $x \in B$  and  $x \in C$  by definition of intersection. By definition of union,  $x \in A \cup B$  and  $x \in A \cup C$ . Thus,  $x \in (A \cup B) \cap (A \cup C)$ .
- $(\supseteq)$ : Let  $x \in (A \cup B) \cap (A \cup C)$ . By definition of intersection,  $x \in A \cup B$  and  $x \in A \cup C$ .
  - If  $x \in A$ , then  $x \in A \cup (B \cap C)$  by definition of union.
  - If  $x \notin A$ , then since  $x \in A \cup B$ , we must have  $x \in B$ . Similarly, since  $x \in A \cup C$ , we must have  $x \in C$ . Since  $x \in B$  and  $x \in C$ ,  $x \in B \cap C$ , and consequently  $x \in A \cup (B \cap C)$ .

Therefore,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

**(6) Distributivity of Intersection over Union:**  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  *Proof:*

- $(\subseteq)$ : Let  $x \in A \cap (B \cup C)$ . By definition of intersection,  $x \in A$  and  $x \in B \cup C$ . By definition of union,  $x \in B$  or  $x \in C$ .
  - If  $x \in B$ , then  $x \in A \cap B$ , and so  $x \in (A \cap B) \cup (A \cap C)$ .
  - If  $x \in C$ , then  $x \in A \cap C$ , and so  $x \in (A \cap B) \cup (A \cap C)$ .

In both cases,  $x \in (A \cap B) \cup (A \cap C)$ .

- $(\supseteq)$ : Let  $x \in (A \cap B) \cup (A \cap C)$ . By definition of union,  $x \in A \cap B$  or  $x \in A \cap C$ .
  - If  $x \in A \cap B$ , then  $x \in A$  and  $x \in B$ . Since  $x \in B$ ,  $x \in B \cup C$ . Thus,  $x \in A \cap (B \cup C)$ .
  - If  $x \in A \cap C$ , then  $x \in A$  and  $x \in C$ . Since  $x \in C$ ,  $x \in B \cup C$ . Thus,  $x \in A \cap (B \cup C)$ .

Therefore,  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

**(7) De Morgan's Law 1:**  $(A \cup B)' = A' \cap B'$  *Proof:*

- $(\subseteq)$ : Let  $x \in (A \cup B)'$ . By definition of complement,  $x \notin A \cup B$ . This means  $x \notin A$  and  $x \notin B$ . By definition of complement,  $x \in A'$  and  $x \in B'$ , which means  $x \in A' \cap B'$  by definition of intersection.
- $(\supseteq)$ : Let  $x \in A' \cap B'$ . By definition of intersection and complement,  $x \notin A$  and  $x \notin B$ . Since  $x$  is neither in  $A$  nor in  $B$ ,  $x \notin A \cup B$ . Thus,  $x \in (A \cup B)'$ .

Therefore,  $(A \cup B)' = A' \cap B'$ .

**(8) De Morgan's Law 2:**  $(A \cap B)' = A' \cup B'$  *Proof:*

- $(\subseteq)$ : Let  $x \in (A \cap B)'$ . By definition of complement,  $x \notin A \cap B$ . This means  $x \notin A$  or  $x \notin B$ . If  $x \notin A$ , then  $x \in A'$ , so  $x \in A' \cup B'$ . If  $x \notin B$ , then  $x \in B'$ , so  $x \in A' \cup B'$ . In either case,  $x \in A' \cup B'$ .
- $(\supseteq)$ : Let  $x \in A' \cup B'$ . By definition of union and complement,  $x \notin A$  or  $x \notin B$ . This is the logical negation of  $(x \in A \wedge x \in B)$ , so  $x \notin A \cap B$ . Therefore,  $x \in (A \cap B)'$ .

Therefore,  $(A \cap B)' = A' \cup B'$ .

**(9) Complementation Law 1:**  $A \cup A' = U$  *Proof:*

- $(\subseteq)$ : Since  $A \subseteq U$  and  $A' \subseteq U$ , their union  $A \cup A'$  is a subset of  $U$ .
- $(\supseteq)$ : Let  $x \in U$ . Either  $x \in A$  or  $x \notin A$ . If  $x \in A$ , then  $x \in A \cup A'$ . If  $x \notin A$ , then by definition of complement  $x \in A'$ , so  $x \in A \cup A'$ .

Therefore,  $A \cup A' = U$ .

**(10) Complementation Law 2:**  $A \cap A' = \emptyset$  *Proof:* Let  $U$  be a universal set with  $A \subseteq U$ . By definition,  $A' = \{x \in U \mid x \notin A\}$ . The intersection  $A \cap A'$  consists of all elements  $x$  such that  $x \in A$  and  $x \in A'$ . However, by definition of  $A'$ , no element  $x \in A$  can belong to  $A'$ . Thus,  $A \cap A'$  contains no elements, so  $A \cap A' = \emptyset$ .

**(11) Self-Inverse Law:**  $(A')' = A$  *Proof:* Let  $U$  be a universal set with  $A \subseteq U$ . By definition,  $A' = \{x \in U \mid x \notin A\}$ . Taking the complement again gives  $(A')' = \{x \in U \mid x \notin A'\}$ . An element  $x \in U$  satisfies  $x \notin A'$  if and only if it is false that  $x \notin A$ , which means  $x \in A$ .

Therefore,  $(A')' = A$ .