

Assignment 8 Solutions

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Introduction to Mathematical Thinking

Answer 1 Claim holds. Let $m = 0$ and $n = 0$. Then:

$$m^2 + mn + n^2 = 0^2 + 0 \cdot 0 + 0^2 = 0 = 0^2$$

Since $0 = 0^2$ is a perfect square, there exist integers m, n such that $m^2 + mn + n^2$ is a perfect square.

Answer 2 Proof: Let m be any positive integer. Choose $n = m + 2$, which is also a positive integer. Then:

$$mn + 1 = m(m + 2) + 1 = m^2 + 2m + 1 = (m + 1)^2$$

Since $m + 1$ is an integer, $(m + 1)^2$ is a perfect square. Therefore, for any positive integer m , there exists a positive integer n such that $mn + 1$ is a perfect square.

Answer 3 Proof: Consider the quadratic polynomial $f(n) = n^2 + 5n + 6$, where $b = 5$ and $c = 6$ are positive integer coefficients.

Factoring $f(n)$ yields:

$$f(n) = (n + 2)(n + 3)$$

For any positive integer $n \geq 1$, we have $n + 2 \geq 3 > 1$ and $n + 3 \geq 4 > 1$. Because $f(n)$ is expressed as the product of two integers strictly greater than 1, $f(n)$ is composite for all positive integers n .

Answer 4 Proof: Assume that every even natural number greater than 2 is a sum of two primes.

Let n be an odd natural number greater than 5. Since n is odd, $n - 3$ is an even number. Furthermore, since $n > 5$, we have $n - 3 > 2$.

By our assumption, the even number $n - 3$ can be written as the sum of two primes p_1 and p_2 :

$$n - 3 = p_1 + p_2 \implies n = p_1 + p_2 + 3$$

Since 3 is also a prime number, n is expressed as the sum of three primes p_1, p_2 , and 3. Therefore, if every even natural number greater than 2 is a sum of two primes, then every odd natural number greater than 5 is a sum of three primes.

Answer 5 Proof by Induction: We wish to prove that $\sum_{i=1}^n (2i - 1) = n^2$ for all $n \geq 1$.

Base Case ($n = 1$): For $n = 1$, the left side is $2(1) - 1 = 1$, and the right side is $1^2 = 1$. The base case holds.

Inductive Step: Assume the statement holds for $n = k$, that is:

$$\sum_{i=1}^k (2i - 1) = k^2$$

We must show it holds for $n = k + 1$:

$$\sum_{i=1}^{k+1} (2i - 1) = \left(\sum_{i=1}^k (2i - 1) \right) + (2(k + 1) - 1)$$

By the inductive hypothesis, substitute $\sum_{i=1}^k (2i - 1) = k^2$:

$$\sum_{i=1}^{k+1} (2i - 1) = k^2 + (2k + 1) = (k + 1)^2$$

Therefore, by the principle of mathematical induction, the sum of the first n odd numbers is n^2 for all $n \geq 1$.

Answer 6 Proof by Induction: We wish to prove that $\sum_{r=1}^n r^2 = \frac{1}{6}n(n + 1)(2n + 1)$ for all $n \geq 1$.

Base Case ($n = 1$):

$$\sum_{r=1}^1 r^2 = 1^2 = 1 \quad \text{and} \quad \frac{1}{6}(1)(1 + 1)(2(1) + 1) = \frac{1}{6}(1)(2)(3) = 1$$

The base case holds.

Inductive Step: Assume the statement holds for $n = k$, that is:

$$\sum_{r=1}^k r^2 = \frac{1}{6}k(k + 1)(2k + 1)$$

We show it holds for $n = k + 1$:

$$\begin{aligned}
\sum_{r=1}^{k+1} r^2 &= \left(\sum_{r=1}^k r^2 \right) + (k+1)^2, \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2, \text{ by the induction hypothesis} \\
&= (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right] \\
&= (k+1) \left[\frac{2k^2 + k + 6k + 6}{6} \right] \\
&= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]
\end{aligned}$$

Therefore, by mathematical induction, $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

ANSWERS TO OPTIONAL PROBLEMS

Answer 1 Direct Proof: Let $N = 1 + 2 + 3 + \cdots + n$. By the commutativity of addition, we can reverse the order of the summation:

$$N = 1 + 2 + 3 + \cdots + n \tag{1}$$

$$N = n + (n-1) + (n-2) + \cdots + 1 \tag{2}$$

Adding equations (1) and (2) term-by-term yields:

$$2N = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

Since there are n such terms in the sum:

$$2N = n(n+1) \implies N = \frac{1}{2}n(n+1)$$

Thus, $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$.

Answer 2 Proof: Let S be a finite collection of points in the plane, not all collinear. Consider the set T of all non-degenerate triangles formed by triples of distinct points in S .

Since S is non-collinear, at least one non-degenerate triangle exists, so T is non-empty. Furthermore, because S is finite, T contains only finitely many triangles.

Each triangle in T has a strictly positive area. Since any finite set of positive real numbers contains a minimum, there exists a triangle $\triangle ABC \in T$ with the minimum possible area.

We claim $\triangle ABC$ contains no other point of S in its interior. Assume for contradiction that a point $P \in S$ lies strictly inside $\triangle ABC$. Connecting P to vertices A, B, C partitions

$\triangle ABC$ into three smaller non-degenerate triangles: $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$. The sum of their areas equals the area of $\triangle ABC$:

$$\text{Area}(\triangle PAB) + \text{Area}(\triangle PBC) + \text{Area}(\triangle PCA) = \text{Area}(\triangle ABC)$$

Since all three sub-triangles have strictly positive area, each must have an area strictly less than $\text{Area}(\triangle ABC)$, contradicting the minimality of $\text{Area}(\triangle ABC)$. Thus, $\triangle ABC$ contains no points of S in its interior.

Answer 3

(a) **Proof by Induction:** We prove $3 \mid (4^n - 1)$ for all $n \geq 1$.

Base Case ($n = 1$): $4^1 - 1 = 3$, which is divisible by 3.

Inductive Step: Assume $4^k - 1$ is divisible by 3 for $n = k$, so $4^k - 1 = 3m$ for some $m \in \mathbb{N} \cup \{0\}$.

For $n = k + 1$:

$$4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(3m + 1) - 1 = 12m + 4 - 1 = 12m + 3 = 3(4m + 1)$$

Since $4m + 1 \in \mathbb{Z}$, $4^{k+1} - 1$ is divisible by 3. By mathematical induction, $3 \mid (4^n - 1)$ for all $n \geq 1$.

(b) **Proof by Induction:** We prove $(n + 1)! > 2^{n+3}$ for all $n \geq 5$.

Base Case ($n = 5$):

$$(5 + 1)! = 6! = 720 \quad \text{and} \quad 2^{5+3} = 2^8 = 256$$

Since $720 > 256$, the base case holds.

Inductive Step: Assume that $(k + 1)! > 2^{k+3}$ holds for an integer $k \geq 5$.

We must show that the statement holds for $n = k + 1$, which means proving:

$$(k + 2)! > 2^{k+4}$$

Starting from the left-hand side, we expand the factorial:

$$(k + 2)! = (k + 2) \cdot (k + 1)!$$

By the inductive hypothesis, $(k + 1)! > 2^{k+3}$. Substituting this into the product yields:

$$(k + 2) \cdot (k + 1)! > (k + 2) \cdot 2^{k+3}$$

Since $k \geq 5$, we know $k + 2 \geq 7 > 2$. Replacing $(k + 2)$ with 2 gives a strictly smaller lower bound:

$$(k + 2) \cdot 2^{k+3} > 2 \cdot 2^{k+3}$$

$$2 \cdot 2^{k+3} = 2^{k+4} = 2^{(k+1)+3}$$

Combining this chain of inequalities:

$$(k+2)! = (k+2)(k+1)! > (k+2) \cdot 2^{k+3} > 2 \cdot 2^{k+3} = 2^{(k+1)+3}$$

Therefore, $(k+2)! > 2^{(k+1)+3}$. By the principle of mathematical induction, $(n+1)! > 2^{n+3}$ holds for all $n \geq 5$.

(c) **Proof by Induction:** We prove $\sum_{r=1}^n r \cdot r! = (n+1)! - 1$ for all $n \geq 1$.

Base Case ($n = 1$):

$$\sum_{r=1}^1 r \cdot r! = 1 \cdot 1! = 1 \quad \text{and} \quad (1+1)! - 1 = 2! - 1 = 1$$

The base case holds.

Inductive Step: Assume $\sum_{r=1}^k r \cdot r! = (k+1)! - 1$ for $n = k$.

For $n = k+1$:

$$\begin{aligned} \sum_{r=1}^{k+1} r \cdot r! &= \left(\sum_{r=1}^k r \cdot r! \right) + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)!, \text{ by the induction hypothesis} \\ &= (k+1)![1 + (k+1)] - 1 \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1 = [(k+1) + 1]! - 1 \end{aligned}$$

By mathematical induction, $\sum_{r=1}^n r \cdot r! = (n+1)! - 1$ for all $n \geq 1$.