

Assignment 7 Solutions

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Introduction to Mathematical Thinking

Answer 1 False. A chicken is a bird and it cannot fly; therefore, the statement “All birds can fly” is false.

Answer 2 False. Let $x = y$, which means $x - y = 0$. Since $0^2 = 0 \cdot 0 = 0$ and $0 \not> 0$, the claim $(\forall x, y \in \mathbb{R})[(x - y)^2 > 0]$ is false.

Answer 3 Suppose a, b are unequal rationals, which means $a = \frac{p}{q}$ and $b = \frac{r}{s}$ with $p, q, r, s \in \mathbb{Z}$, $q, s \neq 0$, and $ps \neq qr$.

Let c be the average of a and b :

$$c = \frac{\frac{p}{q} + \frac{r}{s}}{2} = \frac{\frac{ps+rq}{qs}}{2} = \frac{ps + rq}{2qs}$$

Since $ps + rq \in \mathbb{Z}$ and $2qs \in \mathbb{Z}$ (with $2qs \neq 0$), c is rational.

Without loss of generality, assume $a < b$. Then:

$$\frac{p}{q} < \frac{ps + rq}{2qs} < \frac{r}{s}$$

Therefore, between any two unequal rationals there is a third rational.

Answer 4 Proving $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$ establishes the truth of $\phi \Leftrightarrow \psi$ because $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$ and $\phi \Leftrightarrow \psi$ have identical truth values in a truth table.

Answer 5 Proving $\phi \Rightarrow \psi$ and $(\neg\phi) \Rightarrow (\neg\psi)$ establishes the truth of $\phi \Leftrightarrow \psi$ because $(\phi \Rightarrow \psi) \wedge (\neg\phi \Rightarrow \neg\psi)$ and $\phi \Leftrightarrow \psi$ have identical truth values in a truth table.

Answer 6 Assume for the sake of contradiction that five investors split a payout of \$2,000,000 and every investor receives strictly less than \$400,000.

Let a_1, a_2, a_3, a_4, a_5 be the amounts received by each investor respectively. Since $a_i < 400,000$ for each $i \in \{1, 2, 3, 4, 5\}$, we have:

$$\sum_{i=1}^5 a_i < 5 \times 400,000 = 2,000,000$$

This means the total sum paid out is strictly less than \$2,000,000, which contradicts the fact that the full \$2,000,000 payout was split. Therefore, at least one investor must receive at least \$400,000.

Answer 7 Proof by contradiction. Assume $\sqrt{3}$ is rational. Then it can be written as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$.

$$\sqrt{3} = \frac{p}{q} \implies 3 = \frac{p^2}{q^2} \implies 3q^2 = p^2$$

This implies $3 \mid p^2$. Since 3 is prime, $3 \mid p$. Thus, $p = 3m$ for some $m \in \mathbb{Z}$.

Substituting $p = 3m$:

$$3q^2 = (3m)^2 = 9m^2 \implies q^2 = 3m^2$$

This implies $3 \mid q^2$, and since 3 is prime, $3 \mid q$. Thus, $q = 3n$ for some $n \in \mathbb{Z}$.

Since $3 \mid p$ and $3 \mid q$, 3 is a common factor of p and q , contradicting $\gcd(p, q) = 1$. Therefore, $\sqrt{3}$ is irrational.

Answer 8

- (a) If the Yuan rises, the Dollar falls.
- (b) If $-y < -x$, then $x < y$.
- (c) If two triangles have the same area, then they are congruent.
- (d) If the quadratic equation $ax^2 + bx + c = 0$ has a solution, then $b^2 \geq 4ac$.
- (e) Let $ABCD$ be a quadrilateral. If the opposite angles of $ABCD$ are pairwise equal, then the opposite sides are pairwise equal.
- (f) Let $ABCD$ be a quadrilateral. If all four angles of $ABCD$ are equal, then all four sides are equal.
- (g) If $n^2 + 5$ is divisible by 3, then n is not divisible by 3.

Answer 9

- (b) **They are equivalent.**

Forward: Assume $x < y$. By adding $-x$ to both sides, $x + (-x) < y + (-x) \implies 0 < y - x$. Adding $-y$ to both sides yields $0 + (-y) < (-y) + y - x \implies -y < -x$.

Backward: Assume $-y < -x$. Adding y to both sides yields $-y + y < -x + y \implies 0 < -x + y$. Adding x to both sides yields $x < y$.

- (c) **Original statement is true; converse is false.**

Original statement proof: By definition, two triangles T_1 and T_2 are congruent ($T_1 \cong T_2$) if there exists an isometry f such that $f(T_1) = T_2$. Since isometries preserve metric properties including base length b and altitude h , $\text{Area}(T_1) = \frac{1}{2}bh = \text{Area}(T_2)$.

Converse counterexample: Let T_1 be a right triangle with legs 3 and 4 (Area = 6). Let T_2 be a right triangle with legs 2 and 6 (Area = 6). The side lengths of T_1 are $\{3, 4, 5\}$ and T_2 are $\{2, 6, \sqrt{40}\}$. Since the set of side lengths differ, the triangles are not congruent.

(d) **They are equivalent.**

Forward: Assume $b^2 \geq 4ac$. Then $b^2 - 4ac \geq 0$, so $\sqrt{b^2 - 4ac}$ is a real number. By the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ provides at least one real solution to $ax^2 + bx + c = 0$.

Converse: Assume $ax^2 + bx + c = 0$ has a real solution x_0 with $a \neq 0$. Subtracting c and dividing by a gives:

$$x_0^2 + \frac{b}{a}x_0 = -\frac{c}{a}$$

Completing the square by adding $\frac{b^2}{4a^2}$ to both sides yields:

$$\left(x_0 + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Because the square of any real number is non-negative, the left side is ≥ 0 . Since $4a^2 > 0$, multiplying both sides by $4a^2$ preserves the inequality:

$$b^2 - 4ac \geq 0 \implies b^2 \geq 4ac$$

(e) **They are equivalent.**

Forward: Assume $AB = CD$ and $BC = DA$. Drawing diagonal AC shows $\triangle ABC \cong \triangle CDA$ by SSS, so $\angle B = \angle D$. Drawing diagonal BD similarly shows $\triangle ABD \cong \triangle CDB$, so $\angle A = \angle C$.

Backward: Assume $\angle A = \angle C$ and $\angle B = \angle D$. Since $\angle A + \angle B + \angle C + \angle D = 360^\circ$, we have $2\angle A + 2\angle B = 360^\circ \implies \angle A + \angle B = 180^\circ$, so $AD \parallel BC$. Similarly, $\angle A + \angle D = 180^\circ \implies AB \parallel CD$. $ABCD$ is a parallelogram, so opposite sides are equal.

(f) **Both original statement and converse are false.**

Original statement counterexample: Consider a rhombus $ABCD$ with side length s and $\angle A = 60^\circ$. Then $\angle C = 60^\circ$ and $\angle B = \angle D = 120^\circ$. All sides are equal, but the angles are not equal ($60^\circ \neq 120^\circ$).

Converse counterexample: Consider a rectangle $ABCD$ with $AB = CD = 10$ and $BC = DA = 5$. All four angles are 90° , but the sides are not all equal ($10 \neq 5$).

(g) **They are equivalent.**

Forward proof: By the Division Algorithm, any integer n can be written as $3k, 3k + 1$, or $3k + 2$. If $3 \nmid n$, then $n = 3k + 1$ or $n = 3k + 2$.

- Case 1 ($n = 3k + 1$): $n^2 + 5 = (3k + 1)^2 + 5 = 9k^2 + 6k + 6 = 3(3k^2 + 2k + 2)$, so $3 \mid (n^2 + 5)$.
- Case 2 ($n = 3k + 2$): $n^2 + 5 = (3k + 2)^2 + 5 = 9k^2 + 12k + 9 = 3(3k^2 + 4k + 3)$, so $3 \mid (n^2 + 5)$.

Backward proof: Assume $3 \mid n$, so $n = 3k$ for $k \in \mathbb{Z}$. Then $n^2 + 5 = (3k)^2 + 5 = 9k^2 + 5 = 3(3k^2 + 1) + 2$. The remainder is 2, so $3 \nmid (n^2 + 5)$.

Answer 10 An integer n is divisible by 12 if and only if n^3 is divisible by 12.

Counterexample: Take $n = 18$. 18 is not divisible by 12 but $18^3 = 5832 = 12 \cdot 486$ is. Thus, $12 \mid n^3$ does not imply that $12 \mid n$.

Answer 11 Let r, s be irrationals.

1. $r + 3$: **Necessarily irrational.**

Proof: Assume $r + 3$ is rational, so $r + 3 = \frac{p}{q}$ for $p, q \in \mathbb{Z}, q \neq 0$. Then $r = \frac{p}{q} - 3 = \frac{p-3q}{q}$, which implies r is rational, a contradiction.

2. $5r$: **Necessarily irrational.**

Proof: Assume $5r$ is rational, so $5r = \frac{a}{b}$ for $a, b \in \mathbb{Z}, b \neq 0$. Then $r = \frac{a}{5b}$, which implies r is rational, a contradiction.

3. $r + s$: **Not necessarily irrational.**

Proof: Let $r = \pi$ and $s = -\pi$. $r + s = \pi + (-\pi) = 0$, which is rational.

4. rs : **Not necessarily irrational.**

Proof: Let $r = s = \sqrt{2}$. Then $rs = (\sqrt{2})^2 = 2$, which is rational.

5. $\sqrt{r}$: **Necessarily irrational.**

Proof: Assume $\sqrt{r}$ is rational, so $\sqrt{r} = \frac{a}{b}$ for $a, b \in \mathbb{Z}, b \neq 0$. Then $r = \frac{a^2}{b^2}$, which implies r is rational, a contradiction.

6. r^s : **Not necessarily irrational.**

Proof: Consider $\sqrt{2}^{\sqrt{2}}$.

- Case 1: If $\sqrt{2}^{\sqrt{2}}$ is rational, fix $r = s = \sqrt{2}$.
- Case 2: If $\sqrt{2}^{\sqrt{2}}$ is irrational, fix $r = \sqrt{2}^{\sqrt{2}}$ and $s = \sqrt{2}$. Then $r^s = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.

Answer 12 Let $m, n \in \mathbb{Z}$.

(a) **If m and n are even, then $m + n$ is even.**

Proof: Let $m = 2a$ and $n = 2b$ for $a, b \in \mathbb{Z}$. Then $m + n = 2a + 2b = 2(a + b)$. By closure of addition, $a + b \in \mathbb{Z}$, so $m + n$ is even.

(b) **If m and n are even, then mn is divisible by 4.**

Proof: Let $m = 2a$ and $n = 2b$ for $a, b \in \mathbb{Z}$. Then $mn = (2a)(2b) = 4ab$. By closure of multiplication, $ab \in \mathbb{Z}$, so mn is divisible by 4.

(c) **If m and n are odd, then $m + n$ is even.**

Proof: Let $m = 2a + 1$ and $n = 2b + 1$ for $a, b \in \mathbb{Z}$. Then $m + n = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1)$. By closure of addition, $a + b + 1 \in \mathbb{Z}$, so $m + n$ is even.

- (d) **If one of m, n is even and the other is odd, then $m + n$ is odd.**

Proof: Assume $m = 2a$ and $n = 2b + 1$ for $a, b \in \mathbb{Z}$. Then $m + n = 2a + 2b + 1 = 2(a + b) + 1$. Setting $c = a + b \in \mathbb{Z}$, $m + n = 2c + 1$, which is odd.

- (e) **If one of m, n is even and the other is odd, then mn is even.**

Proof: Assume $m = 2a$ and $n = 2b + 1$ for $a, b \in \mathbb{Z}$. Then $mn = (2a)(2b + 1) = 4ab + 2a = 2(2ab + a)$. Setting $c = 2ab + a \in \mathbb{Z}$, $mn = 2c$, which is even.

ANSWERS TO OPTIONAL PROBLEMS

Answer 1

- (a) **True.**

Proof: Let y be any real number and $x = 0$. Then $x + y = 0 + y = y$.

- (b) **True.**

Proof: For any x , choose $y = -x$. If $x = 0$, $y = 0 \implies x + y = 0$. If $x \neq 0$, $y = -x \implies x + (-x) = 0$.

- (c) **False.**

Proof: Let $a = 10, b = 2$ and $c = 15$. $10 \mid 30$ but $10 \nmid 2$ and $10 \nmid 15$. Therefore, the statement is false.

- (d) **True.**

Proof: Let $x = \frac{p}{q}$ ($p, q \in \mathbb{Z}, q \neq 0$) and y be irrational. Assume $x + y$ is rational, so $x + y = \frac{r}{s}$ ($r, s \in \mathbb{Z}, s \neq 0$). Then $y = \frac{r}{s} - \frac{p}{q} = \frac{rq - ps}{sq}$, which implies y is rational, contradicting that y is irrational. Thus $x + y$ is irrational.

- (e) **True.**

Proof by Contrapositive: Assume x and y are both rational ($x = \frac{a}{b}, y = \frac{c}{d}$ with $b, d \neq 0$). Then $x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$, which is rational. By contrapositive, if $x + y$ is irrational, then at least one of x, y is irrational.

- (f) **False.**

Proof: Let $x = \pi$ and $y = -\pi + 1$. $x + y = \pi - \pi + 1 = 1$, which is rational but none of x, y is rational.