

Assignment 4 Solutions

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Introduction to Mathematical Thinking

Answer 1

ϕ	ψ	$\phi \Rightarrow \psi$	$\psi \Rightarrow \phi$	$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$	$\phi \Leftrightarrow \psi$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Answer 2

ϕ	ψ	$\neg\phi$	$\phi \Rightarrow \psi$	$\neg\phi \vee \psi$	$(\phi \Rightarrow \psi) \Leftrightarrow (\neg\phi \vee \psi)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Answer 3

ϕ	ψ	$\neg\psi$	$\phi \Rightarrow \psi$	$\phi \not\Rightarrow \psi$	$\phi \wedge \neg\psi$	$(\phi \not\Rightarrow \psi) \Leftrightarrow (\phi \wedge \neg\psi)$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Answer 4 (a)

ϕ	ψ	$\phi \Rightarrow \psi$	$\phi \wedge (\phi \Rightarrow \psi)$	$[\phi \wedge (\phi \Rightarrow \psi)] \Rightarrow \psi$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(b) The modus ponens is a valid rule of inference because its truth values are all T for all ϕ and ψ .

Answer 5

1. $\phi \vee \psi$ means that at least one of ϕ and ψ is true.
2. $\neg(\phi \vee \psi)$ means that it is not the case that at least one of ϕ and ψ is true.
3. If it is not the case that at least one of them is true then ϕ and ψ are both false.
4. This is the same as saying that both $\neg\phi$ and $\neg\psi$ are true (by the definition of negation).
5. By the meaning of and, this can be expressed as $(\neg\phi) \wedge (\neg\psi)$.

Answer 6

- (a) 34,159 is not a prime number.
- (b) Roses are not red or violets are not blue.
- (c) There are no hamburgers, but I won't have a hot dog.
- (d) Fred won't go or he'll play.
- (e) The number x is nonnegative and less than or equal to 10.
- (f) We'll lose the first and second game.

Answer 7

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\phi \Leftrightarrow \psi$	$(\neg\phi) \Leftrightarrow (\neg\psi)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Answer 8 (a)

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

(b)

ϕ	ψ	θ	$\psi \vee \theta$	$\phi \Rightarrow (\psi \vee \theta)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

Answer 9

ϕ	ψ	θ	$\psi \wedge \theta$	$\phi \Rightarrow \psi$	$\phi \Rightarrow \theta$	$\phi \Rightarrow (\psi \wedge \theta)$	$(\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \theta)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Answer 10 $[\Rightarrow]$ Assuming ϕ , $\phi \Rightarrow (\psi \wedge \theta)$ means that $(\psi \wedge \theta)$ is true.

But for $(\psi \wedge \theta)$ to be true, both ψ and θ have to be true (by the definition of and). Thus, whenever ϕ is true, ψ is also true and whenever ϕ is true, θ is also true. By the meaning of and, it can be expressed as $(\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \theta)$.

$[\Leftarrow]$ $(\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \theta)$ means that both $(\phi \Rightarrow \psi)$ and $(\phi \Rightarrow \theta)$ are true.

Assuming ϕ , for both $(\phi \Rightarrow \psi)$ and $(\phi \Rightarrow \theta)$ to be true, both ψ and θ have to be true (by the definition of implication). Thus, if ϕ is true, ψ and θ are true. This can be expressed as $\phi \Rightarrow (\psi \wedge \theta)$.

Answer 11

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\phi \Rightarrow \psi$	$(\neg\psi) \Rightarrow (\neg\phi)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Answer 12

- (a) If two rectangles do not have the same area, they are not congruent.
- (b) If in a triangle with sides a, b, c (c largest), $a^2 + b^2 \neq c^2$, then the triangle is not right-angled.

- (c) If n is not prime, then $2^n - 1$ is not prime.
- (d) If the Dollar doesn't fall, the Yuan won't rise.

Answer 13

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\phi \Rightarrow \psi$	$\neg\psi \Rightarrow \neg\phi$	$\psi \Rightarrow \phi$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Answer 14

- (a) If two rectangles have the same area, they are congruent.
- (b) If in a triangle with sides a, b, c (c largest), $a^2 + b^2 = c^2$, then the triangle is right-angled.
- (c) If n is prime, then $2^n - 1$ is prime.
- (d) If the Dollar falls, the Yuan will rise.

ANSWERS TO OPTIONAL PROBLEMS

Answer 1 $\neg\psi \Rightarrow \phi$

Answer 2

ϕ	ψ	$\phi \dot{\vee} \psi$
T	T	F
T	F	T
F	T	T
F	F	F

Answer 3

ϕ	ψ	$\phi \vee \psi$	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$	$(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Answer 4

- (a) If x is an even integer, then x is divisible by 2.
- (b) If a number is divisible by 6, then it is divisible by 2.
- (c) If a quadrilateral is a square, then it is a rectangle.
- (d) Impossible. A conditional and its contrapositive are logically equivalent and must share the same truth value.

Answer 5

M	N	$M \times N$	$M + N$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

Answer 6

(a) $\wedge$

(b) $\dot{\vee}$

(c) No

Answer 7

M	N	$M \times N$	$M + N$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	T	T

(a) $\vee$

(b) $\Leftrightarrow$

(c) No

Answer 8 The least number of cards you have to turn to verify this rule is 2. The 4 to test the contrapositive and the E to test the Modus Ponens.

Answer 9 Let m, n be natural numbers.

Proposition : mn is odd iff m and n are odd.

[$\Rightarrow$] Let prove that if mn is odd, then m and n are odd.

This is the same as saying that if at least one of m, n is not odd, then mn is not odd.

For a natural number to not be odd it has to be divisible by 2.

Case 1 :

Let $m = 2a, n = 2b$

$mn = 2a \cdot 2b = 2(a \cdot b)$

By the closure property of multiplication on the natural numbers, $a \cdot b \in \mathbb{N}$. So mn is even.

Case 2 :

Let $m = 2a, n = 2b + 1$

$mn = 2a \cdot (2b + 1) = 4ab + 2a = 2(2ab + a)$

By the closure property of multiplication on the natural numbers, $(2ab + a) \in \mathbb{N}$. So mn is even.

Therefore, if mn is odd, then m and n are odd.

[$\Leftarrow$] Let prove that if m and n are odd then mn is odd.

Let $m = 2a + 1$ and $n = 2b + 1$

$$mn = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$$

Let $(2ab + a + b) = c$, with $c \in \mathbb{N}$ $mn = 2c + 1$, which is odd.

Therefore, if m and n are odd then mn is odd and the proposition is true.

Answer 10 Let m, n be natural numbers.

Proposition : mn is even iff m and n are even.

Counter example :

Let $m = 2$ and $n = 5$

$mn = 10$, which is even but one of m, n is odd.

Therefore, the proposition is false.

Answer 11 Check the ID of the one drinking beer to ensure they are legally allowed to drink and check the drink of the one who is under age to make sure it's not alcohol.

Answer 12 The two problems have the same underlying logical structure : a conditional statement of the form "if P , then Q ".

In Wason's problem, the rule is : If there is a vowel on one side, then there is an odd number on the other side.

In the drinking-age problem, the rule is :

If someone is drinking alcohol, then they are above the legal drinking age.

The correspondence between the two problem is :

- Vowel : drinking alcohol
- Consonant : drinking a non-alcoholic drink
- Odd number : above the legal age
- Even number : under the legal age

In solving both problems, I used modus ponens (checking the case where P is true) and modus tollens (checking the case where Q is false). This is why, in both problems, the correct checks are the " P " case and the "not Q " case, while the "not P " and " Q " cases are irrelevant.

The drinking-age problem felt easier because it is less abstract and involves familiar social rules.