Assignment 6 Solutions

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Introduction to Mathematical Thinking

Answer 1 To show that $\neg[\exists x A(x)] \iff \forall x[\neg A(x)]$, we must prove the implication in both directions:

- ($\Rightarrow$) Suppose $\neg[\exists x A(x)]$. This statement asserts that it is not the case that there exists an x such that $A(x)$ holds. Consequently, for every x in the domain, $A(x)$ must be false. Thus, $\forall x[\neg A(x)]$ holds.
- ($\Leftarrow$) Suppose $\forall x[\neg A(x)]$. This means that for every x in the domain, $A(x)$ is false. If there were even a single x such that $A(x)$ were true, it would contradict our hypothesis that it is false for all x. Therefore, no such x exists, which is symbolized by $\neg[\exists x A(x)]$.

Since both directions hold, the equivalence is proven.

Answer 2 To prove the statement is false, we examine the definition of prime and even numbers. By definition, a prime number p is a natural number greater than 1 whose only divisors are 1 and p. A positive even number n is defined as $n = 2k$ for some $k \in \mathbb{N}$.

If there were an even prime $q > 2$, then $q = 2k$ for some $k > 1$. This would mean that q has at least three distinct divisors: 1, 2, and q itself (since $k > 1$ implies $2k > 2$). This contradicts the definition of a prime number. Thus, no such even prime exists.

Answer 3

(a) Let P be the set of all people. Let $S(x)$ mean x is a student. Let $Pz(x)$ mean x likes pizza.
$(\forall x \in P)[S(x) \implies Pz(x)]$

(b) Let P be the set of all people. Let $F(x)$ mean x is my friend. Let $C(x)$ mean x has a car.
$(\exists x \in P)[F(x) \wedge \neg C(x)]$

(c) Let A be the set of all animals. Let $E(x)$ mean x is an elephant. Let $M(x)$ mean x likes muffins.
$(\exists x \in A)[E(x) \wedge \neg M(x)]$

(d) Let G be the set of all geometric figures. Let $T(x)$ mean x is a triangle. Let $I(x)$ mean x is isosceles.
$(\forall x \in G)[T(x) \implies I(x)]$

(e) Let P be the set of all people. Let $S(x)$ mean x is a student. Let $T(x)$ mean x is here today.
$(\exists x \in P)[S(x) \wedge \neg T(x)]$

(f) Let P be the set of all people. Let $L(x, y)$ mean x loves y.
$(\forall x \in P)(\exists y \in P)[L(x, y)]$

(g) Let P be the set of all people. Let $L(x, y)$ mean x loves y.
$(\forall x \in P)(\exists y \in P)[\neg L(x, y)]$

(h) Let P be the set of all people. Let $M(x)$ mean x is a man. Let $C(x)$ mean x comes. Let $W(x)$ mean x is a woman. Let $L(x)$ mean x leaves.
$[(\exists x \in P)(M(x) \wedge C(x))] \implies [(\forall y \in P)(W(y) \implies L(y))]$

(i) Let P be the set of all people. Let $T(x)$ mean x is tall. Let $S(x)$ mean x is short.
$(\forall x \in P)[T(x) \vee S(x)]$

(j) Let P be the set of all people. Let $T(x)$ mean x is tall. Let $S(x)$ mean x is short.
$(\forall x \in P)[T(x)] \vee (\forall y \in P)[S(y)]$

(k) Let S be the set of all stones. Let $P(x)$ mean x is precious. Let $B(x)$ mean x is beautiful.
$(\exists x \in S)[P(x) \wedge \neg B(x)]$

(l) Let P be the set of all people. Let $L(x)$ mean x loves me.
$(\forall x \in P)[\neg L(x)]$

(m) Let S be the set of all snakes. Let $A(x)$ mean x is American. Let $P(x)$ mean x is poisonous.
$(\exists x \in S)[A(x) \wedge P(x)]$

(n) Let An be the set of all animals. Let $S(x)$ mean x is a snake. Let $A(x)$ mean x is American. Let $P(x)$ mean x is poisonous.
$(\exists x \in An)[S(x) \wedge A(x) \wedge P(x)]$

Answer 4

(a) $(\exists x \in P)[S(x) \wedge \neg Pz(x)]$ — Not all students like pizza.

(b) $(\forall x \in P)[F(x) \implies C(x)]$ — All my friends have a car.

(c) $(\forall x \in A)[E(x) \implies M(x)]$ — All elephants like muffins.

(d) $(\exists x \in G)[T(x) \wedge \neg I(x)]$ — Not all triangles are isosceles.

(e) $(\forall x \in P)[S(x) \implies T(x)]$ — All students in the class are here today.

(f) $(\exists x \in P)(\forall y \in P)[\neg L(x, y)]$ — Some people don't love anyone.

(g) $(\exists x \in P)(\forall y \in P)[L(x, y)]$ — Some people love everybody.

(h) $[(\exists x \in P)(M(x) \wedge C(x))] \wedge [(\exists y \in P)(W(y) \wedge \neg L(y))]$ — A man comes, and yet some woman does not leave.

(i) $(\exists x \in P)[\neg T(x) \wedge \neg S(x)]$ — Some people are neither tall nor short.

(j) $(\exists x \in P)[\neg T(x)] \wedge (\exists y \in P)[\neg S(y)]$ — Some people are not tall and some people are not short.

(k) $(\forall x \in S)[P(x) \implies B(x)]$ — All precious stones are beautiful.

(l) $(\exists x \in P)[L(x)]$ — Someone loves me.

(m) $(\forall x \in S)[A(x) \implies \neg P(x)]$ — All American snakes are not poisonous.

(n) $(\forall x \in An)[(S(x) \wedge A(x)) \implies \neg P(x)]$ — All American snakes are not poisonous.

Answer 5

(a) False

(b) False

(c) True

(d) True

(e) False

(f) False

(g) False

(h) True

Answer 6

(a) $\forall x(2x + 3 \neq 5x + 1)$

(b) $\forall x(x^2 \neq 2)$

(c) $\exists x \forall y(y \neq x^2)$

(d) $\exists x \forall y(y \neq x^2)$

(e) $\exists x \forall y \exists z(xy \neq xz)$

(f) $\exists x \forall y \exists z(xy \neq xz)$

(g) $\exists x[x < 0 \wedge \forall y(y^2 \neq x)]$

(h) $\exists x[x < 0 \wedge \forall y(y^2 \neq x)]$

Answer 7

(a) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(x + y \neq 1)$

(b) $(\exists x > 0)(\forall y < 0)(x + y \neq 0)$

(c) $\forall x(\exists \epsilon > 0)(x \leq -\epsilon \vee x \geq \epsilon)$

(d) $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})(\forall z \in \mathbb{N})(x + y \neq z^2)$

Answer 8 $(\forall t)(\exists p)[\neg F(p,t)] \vee (\forall p)(\exists t)[\neg F(p,t)] \vee [\forall p \forall t F(p,t)]$

Answer 9 $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x)[|x - a| < \delta \wedge |f(x) - f(a)| \geq \epsilon]$